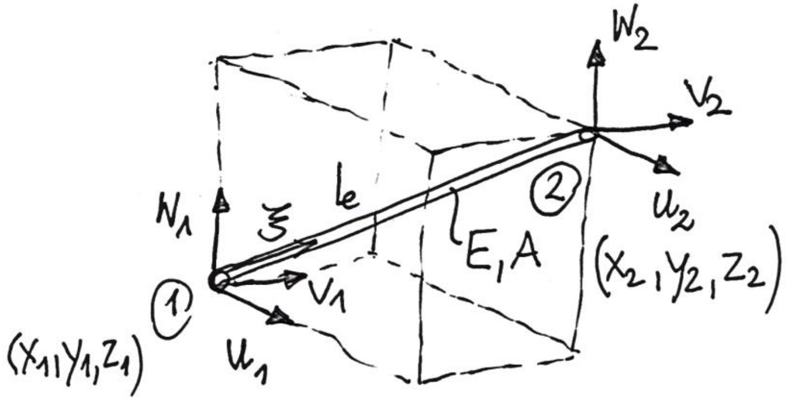
Politechnika Warszawska | Wydział Mechaniczny Energetyki i Lotnictwa | Zakład Wytrzymałości Materiałów i Konstrukcji

Metoda elementów skończonych (MES1)

Wykład 8A. 3D element kratownicy

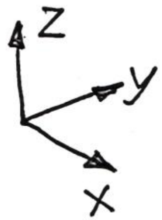
04.2022

Element skończony pręta kratownicy 3D



$$\underset{6\times 1}{\{q_g\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}_e$$

Globalny wektor parametrów węzłowych

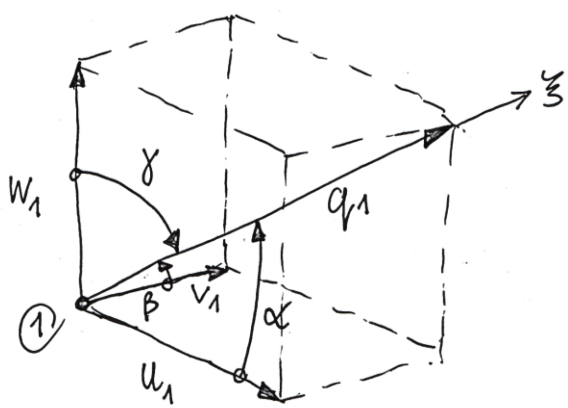


$$l_e = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Lokalny wektor parametrów węzłowych

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$

Przemieszczenia węzła 1

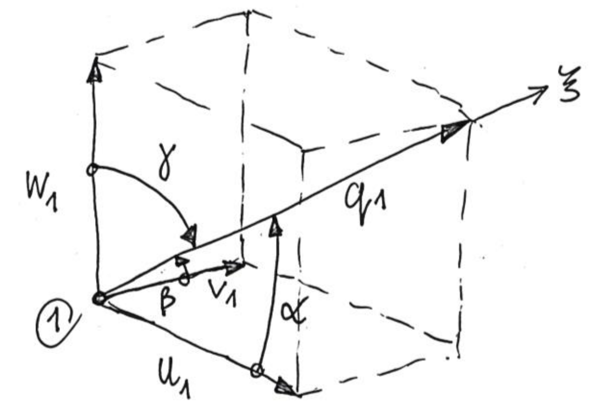


$$\cos\alpha = \frac{x_2 - x_1}{l_e}$$

$$\cos\beta = \frac{y_2 - y_1}{l_e}$$

$$\cos\gamma = \frac{z_2 - z_1}{l_e}$$

Przemieszczenia węzła 1



$$u_1 = q_1 \cos\alpha$$
$$v_1 = q_1 \cos\beta$$
$$w_1 = q_1 \cos\gamma$$

$$u_1 \cos\alpha = q_1 \cos^2\alpha$$
$$v_1 \cos\beta = q_1 \cos^2\beta$$
$$w_1 \cos\gamma = q_1 \cos^2\gamma$$

$$u_1 \underbrace{\cos\alpha}_{a} + v_1 \underbrace{\cos\beta}_{b} + w_1 \underbrace{\cos\gamma}_{c} = q_1$$

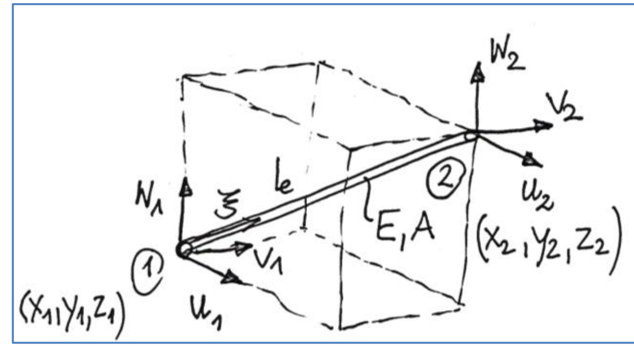
Przemieszczenia węzła 1 i 2

$$q_1 = a \cdot u_1 + b \cdot v_1 + c \cdot w_1 + 0 \cdot u_2 + 0 \cdot v_2 + 0 \cdot w_2$$
$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + 0 \cdot w_1 + a \cdot u_2 + b \cdot v_2 + c \cdot w_2$$

Wektor przemieszczeń węzłowych elementu kratownicy

$$q_1 = a \cdot u_1 + b \cdot v_1 + c \cdot w_1 + 0 \cdot u_2 + 0 \cdot v_2 + 0 \cdot w_2$$

$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + 0 \cdot w_1 + a \cdot u_2 + b \cdot v_2 + c \cdot w_2$$



$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \begin{bmatrix} a & b & c & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}_e$$

$$\underset{2\times 1}{\{q\}_e} = \underset{2\times 6}{[T_t]_e} \cdot \underset{6\times 1}{\{q_g\}_e}$$

$$\underset{1\times 2}{\lfloor q \rfloor_e} = \underset{1\times 6}{\lfloor q_g \rfloor} \cdot \underset{6\times 2}{[T_t]_e^T} \quad ; \quad [T_t]_e^T = \begin{bmatrix} a & 0 \\ b & 0 \\ c & 0 \\ 0 & a \\ 0 & b \\ 0 & c \end{bmatrix}$$

Macierz transformacji

Energia sprężysta elementu kratownicy:

$$U_e = \frac{1}{2}\underset{1\times 2}{\lfloor q \rfloor_e} \cdot \underset{2\times 2}{[k]_e} \cdot \underset{2\times 1}{\{q\}_e} = \frac{1}{2}\underset{1\times 6}{\lfloor q_g \rfloor_e} \cdot \underset{6\times 2}{[T_t]_e^T} \underset{2\times 2}{[k]_e} \underset{2\times 6}{[T_t]_e} \cdot \underset{6\times 1}{\{q_g\}_e} =$$

$$= \frac{1}{2}\underset{1\times 6}{\lfloor q_g \rfloor_e} \underset{6\times 6}{[k_g]_e} \cdot \underset{6\times 1}{\{q_g\}_e}$$, gdzie:

$$\underset{2\times 2}{[k]_e} = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Globalna macierz sztywności
pręta kratownicy:

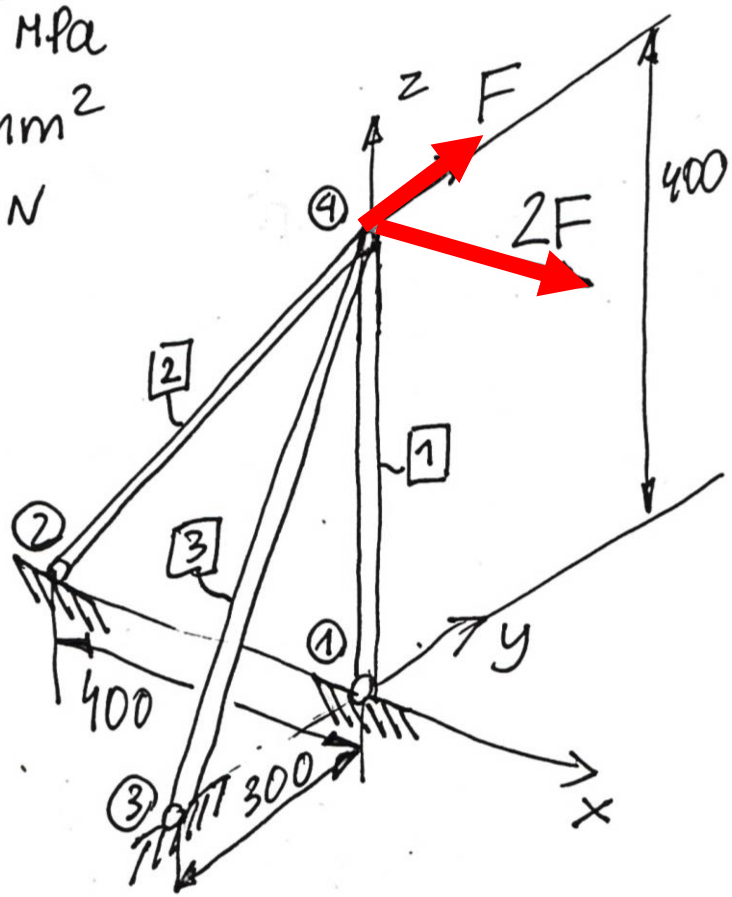
$$\underset{6\times 6}{[k_g]_e} = \frac{EA}{l_e}\begin{bmatrix} a^2 & ab & ac & -a^2 & -ab & -ac \\ ab & b^2 & bc & -ab & -b^2 & -bc \\ ac & bc & c^2 & -ac & -bc & -c^2 \\ -a^2 & -ab & -ac & a^2 & ab & ac \\ -ab & -b^2 & -bc & ab & b^2 & bc \\ -ac & -bc & -c^2 & ac & bc & c^2 \end{bmatrix}$$

Przykład Zbuduj model MES przestrzennej kratownicy.
Znajdź przemieszczenia węzłowe, naprężenia, siły wewnętrzne i reakcje

$E = 2 \cdot 10^5$ MPa

$A = 100 \text{mm}^2$

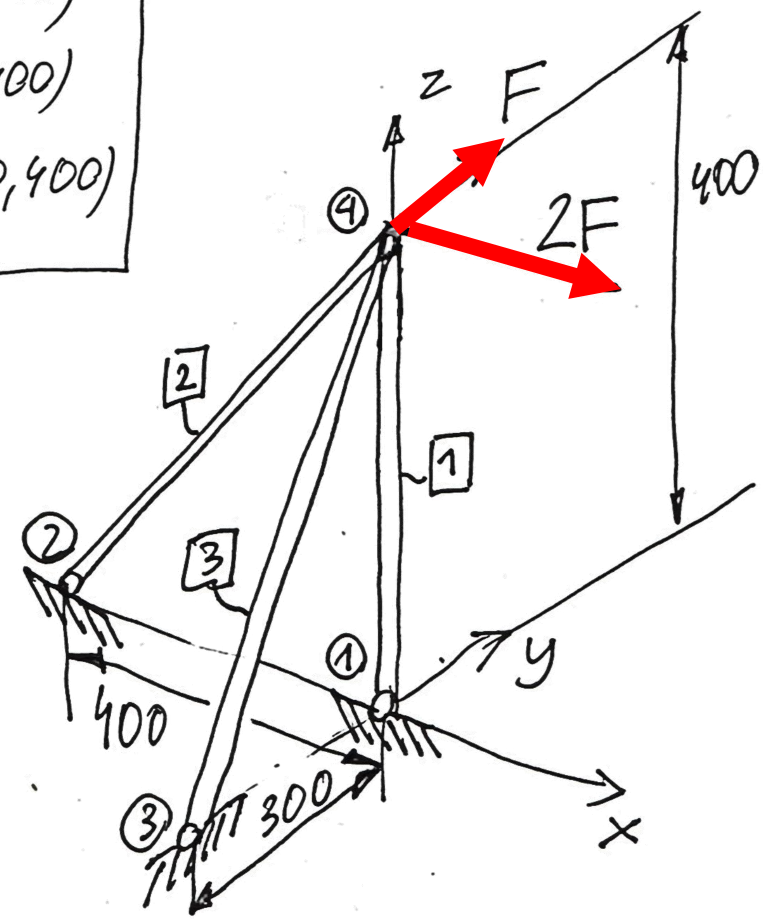
$F = 1500$ N



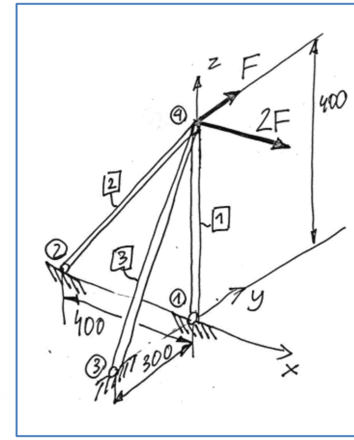
$$\{q\}_{12\times 1} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

Model MES

ELEMENT	NODES
1	① (0, 0, 0) → ④ (0, 0, 400)
2	② (−400, 0, 0) → ④ (0, 0, 400)
3	③ (0, −300, 0) → ④ (0, 0, 400)



Macierz sztywności elementu 1

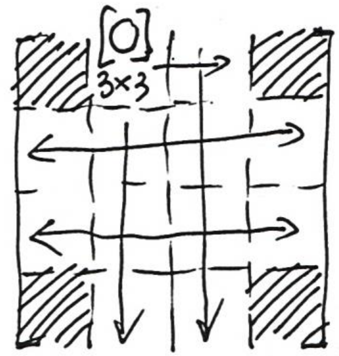


ELEMENT $\boxed{1}$; $l_1 = 400\,mm$

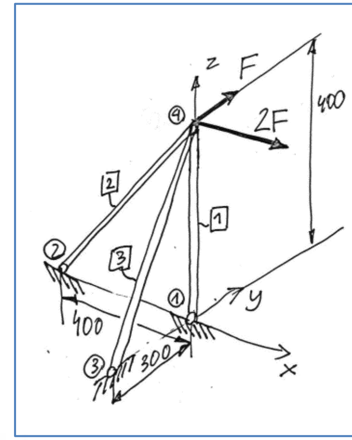
$$a_1 = \frac{0-0}{l_1} = 0 \quad ; \quad b_1 = \frac{0-0}{l_1} = 0, \quad c_1 = \frac{400-0}{400} = 1$$

$$\underset{2\times 6}{[T_t]_1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_1} = \frac{EA}{l_1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underset{12\times 12}{[k_g]_1^*} =$$



Macierz sztywności elementu 2



ELEMENT $\boxed{2}$

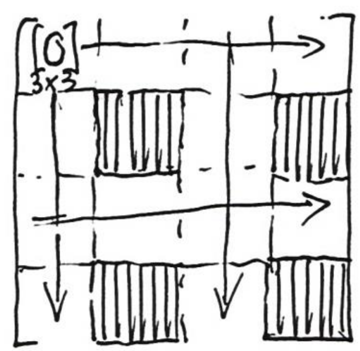
$$l_2 = \sqrt{(0-(-400))^2 + (0-0)^2 + (400-0)^2} = 400\sqrt{2}\ mm$$

$$a_2 = \frac{0-(-400)}{400\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad b_2 = \frac{0-0}{400\sqrt{2}} = 0, \quad c_2 = \frac{400-0}{400\sqrt{2}} = \frac{\sqrt{2}}{2}$$

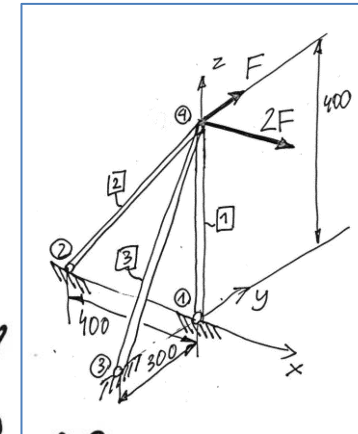
$$\underset{2\times 6}{[T_t]_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_2} = \frac{EA}{l_2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\underset{12\times 12}{[K_g]_2^*} =$$



Macierz sztywności elementu 3



ELEMENT $\boxed{3}$

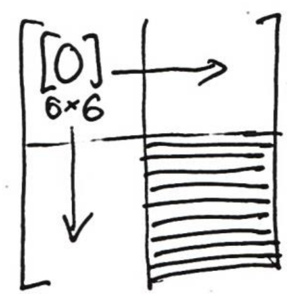
$$l_3 = \sqrt{(0-0)^2 + (0-(-300))^2 + (400-0)^2} = 500\,mm$$

$$a_3 = \frac{0-0}{500} = 0\,, \quad b_3 = \frac{0-(-300)}{500} = \frac{3}{5} = 0.6\,, \quad c_3 = \frac{400-0}{500} = 0.8$$

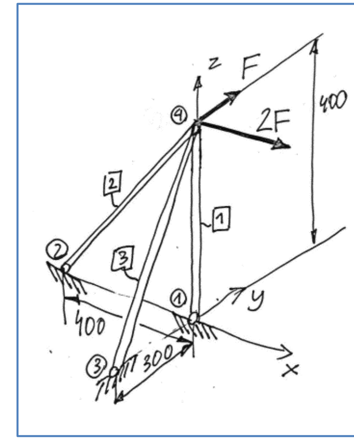
$$\underset{2\times 6}{[T_t]_3} = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_3} = \frac{EA}{l_3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.36 & 0.48 & 0 & -0.36 & -0.48 \\ 0 & 0.48 & 0.64 & 0 & -0.48 & -0.64 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.36 & -0.48 & 0 & 0.36 & 0.48 \\ 0 & -0.48 & -0.64 & 0 & 0.48 & 0.64 \end{bmatrix}$$

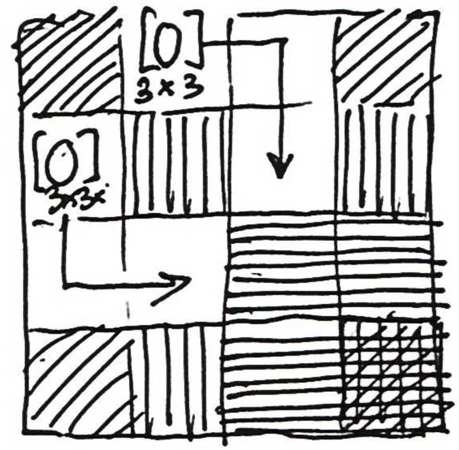
$$\underset{12\times 12}{[k_g]_3^*} =$$



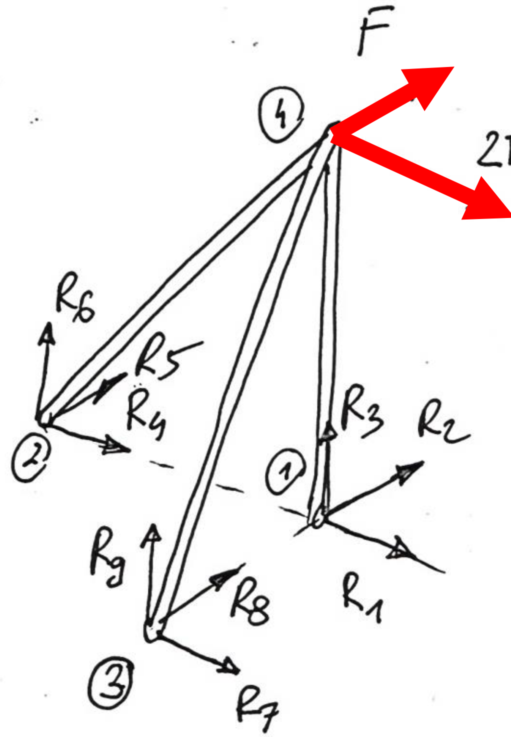
Globalna macierz sztywności modelu kratownicy



$$\underset{12\times12}{[K]} = [k_g]_1^* + [k_g]_2^* + [k_g]_3^* =$$

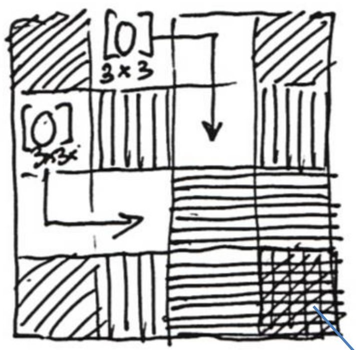


Globalny wektor obciążenia:



$$\underset{12\times1}{\{F\}}=\begin{Bmatrix}R_1\\R_2\\R_3\\R_4\\R_5\\R_6\\R_7\\R_8\\R_9\\2F\\F\\0\end{Bmatrix}$$

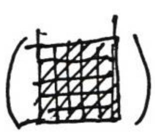
Globalna macierz sztywności:



Warunki brzegowe:

$$u_1=v_1=w_1=u_2=v_2=w_2=u_3=v_3=w_3=0$$

$$N = NDOF - NOF = 12-9=3$$



$$\underset{3\times3}{[K]}\cdot\begin{Bmatrix}u_4\\v_4\\w_4\end{Bmatrix}=\begin{Bmatrix}2F\\F\\0\end{Bmatrix}$$

Układ równań:

$$\underset{3\times 3}{[K]} = EA \begin{bmatrix} \frac{1}{2l_2} & 0 & \frac{1}{2l_2} \\ 0 & \frac{0.36}{l_3} & \frac{0.48}{l_3} \\ \frac{1}{2l_2} & \frac{0.48}{l_3} & \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right) \end{bmatrix}$$

$$\underset{3\times 3}{[K]} \cdot \begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 2F \\ F \\ 0 \end{Bmatrix}$$

$$\begin{cases} \frac{1}{2l_2} \cdot u_4 + 0 \cdot v_4 + \frac{1}{2l_2} w_4 = \frac{2F}{EA} & \Big| \cdot 2l_2 \\ 0 \cdot u_4 + \frac{0.36}{l_3} v_4 + \frac{0.48}{l_3} w_4 = \frac{F}{EA} & \Big| \cdot \frac{100\, l_3}{36} \\ \frac{1}{2l_2} \cdot u_4 + \frac{0.48}{l_3} v_4 + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right) w_4 = 0 \end{cases}$$

$$
\begin{cases}
u_4 + w_4 = \dfrac{4Fl_2}{EA} & \Rightarrow \quad u_4 = \dfrac{4Fl_2}{EA} - w_4 \\[2ex]
v_4 + \dfrac{4}{3}w_4 = \dfrac{100\,Fl_3}{36\,EA} & \Rightarrow \quad v_4 = \dfrac{100\,Fl_3}{36\,EA} - \dfrac{4}{3}w_4 \\[2ex]
\dfrac{1}{2l_2}u_4 + \dfrac{0.48}{l_3}v_4 + \left(\dfrac{1}{l_1} + \dfrac{1}{2l_2} + \dfrac{0.64}{l_3}\right)w_4 = 0
\end{cases}
$$

$$\frac{1}{2l_2}\cdot\left(\frac{4Fl_2}{EA} - w_4\right) + \frac{0.48}{l_3}\cdot\left(\frac{100}{36}\frac{Fl_3}{EA} - \frac{4}{3}w_4\right) + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right)w_4 = 0$$

$$\frac{2F}{EA} + \frac{4F}{3EA} + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3} - \frac{1}{2l_2} - \frac{0.64}{l_3}\right)w_4 = 0$$

$$w_4 = -\left(\frac{2F}{EA} + \frac{4F}{3EA}\right)l_1 = -\frac{10\,Fl_1}{3\,EA} = -0.1\,mm$$

$$u_4 = \frac{4Fl_2}{EA} + \frac{10Fl_1}{3EA} = \frac{12Fl_2}{3EA} + \frac{10Fl_1}{3EA} = \frac{(12l_2 + 10l_1)F}{3EA} = 0.27\,mm$$

$$v_4 = \frac{100\,Fl_3}{36\,EA} - \frac{4}{3}\cdot\left(-\frac{10\,Fl_1}{3\,EA}\right) = \frac{100Fl_3}{36EA} + \frac{40Fl_1}{9\,EA} = \frac{(100l_3 + 160l_1)F}{36EA} = 0.2375\,mm$$

Reakcje:

$$\underset{12\times 12}{[K]} \cdot \underset{12\times 1}{\{q\}} = \underset{12\times 1}{\{F\}}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -50000 \\ 0 & 0 & 0 & 17677.67 & 0 & 17677.67 & 0 & 0 & 0 & -17677.7 & 0 & -17677.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17677.67 & 0 & 17677.67 & 0 & 0 & 0 & -17677.7 & 0 & -17677.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14400 & 19200 & 0 & -14400 & -19200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19200 & 25600 & 0 & -19200 & -25600 \\ 0 & 0 & 0 & -17677.7 & 0 & -17677.7 & 0 & 0 & 0 & 17677.67 & 0 & 17677.67 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14400 & -19200 & 0 & 14400 & 19200 \\ 0 & 0 & -50000 & -17677.7 & 0 & -17677.7 & 0 & -19200 & -25600 & 17677.67 & 19200 & 93277.67 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.269706 \\ 0.2375 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5000 \\ -3000 \\ 0 \\ -3000 \\ 0 \\ -1500 \\ -2000 \\ 3000 \\ 1500 \\ 0 \end{bmatrix}$$

$$R_1 = 0\,,\quad R_2 = 0\,,$$

$$R_3 = -\frac{EA}{l_1}\cdot w_4 = -\frac{EA}{l_1}\cdot\left(-\frac{10}{3}\frac{Fl_1}{EA}\right) = \frac{10}{3}F = 5000\,N$$

$$R_4 = -\frac{EA}{2l_2}\cdot u_4 - \frac{EA}{2l_2}w_4 = -\frac{EAF}{2l_2}\left(\frac{12l_2+10l_1}{3EA} - \frac{10l_1}{3EA}\right) = -2F = -3000\,N$$

$$R_5 = 0\,,\quad R_6 = -\frac{EA}{2l_2}u_4 - \frac{EA}{2l_2}w_4 = R_4 = -2F = -3000\,N$$

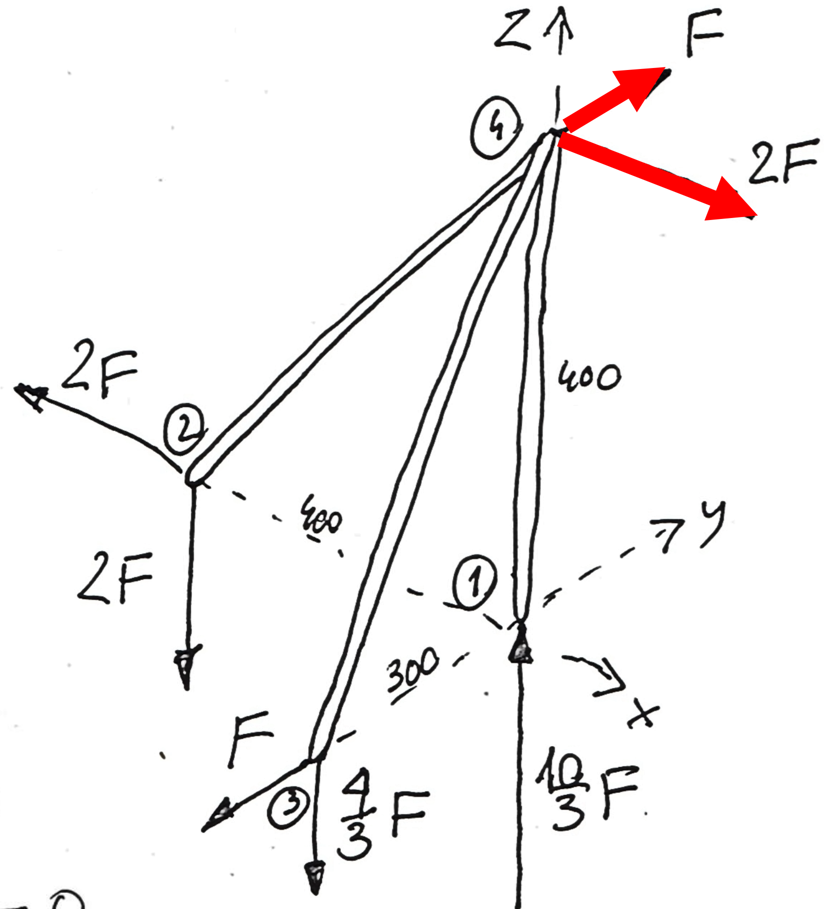
$$R_7 = 0\,,\quad R_8 = -\frac{0.36EA}{l_3}\cdot v_4 - \frac{0.48EA}{l_3}w_4 =$$

$$= -\frac{0.36EA}{l_3}\cdot\frac{100(l_3+1.6l_1)F}{36EA} + \frac{0.48EA}{l_3}\cdot\frac{10}{3}\frac{Fl_1}{EA} = -F = -1500\,N$$

$$R_9 = \frac{-0.48EA}{l_3}\cdot v_4 - \frac{0.64EA}{l_3}w_4 =$$

$$= -\frac{0.48EA}{l_3}\cdot\frac{100(l_3+1.6l_1)F}{36EA} + \frac{0.64\cdot EA}{l_3}\cdot\frac{10}{3}\frac{Fl_1}{EA} = -\frac{4F}{3} = -2000\,N$$

Równowaga sił:



$$\sum F_x = 0 : \quad -2F + 2F = 0$$

$$\sum F_y = 0 : \quad -F + F = 0$$

$$\sum F_z = 0 : \quad \frac{10}{3}F - 2F - \frac{4}{3}F = 0$$

$$\sum M_x^{(1)} = 0 : \quad \frac{4}{3}F \cdot 300\,\mathrm{mm} - F \cdot 400\,\mathrm{mm} = 0$$

$$\sum M_y^{(1)} = 0 : \quad -2F \cdot 400\,\mathrm{mm} + 2F \cdot 400\,\mathrm{mm} = 0$$

$$\sum M_z^{(2)} = 0 : \quad -F \cdot 400\,\mathrm{mm} + F \cdot 400\,\mathrm{mm} = 0$$

Rozwiązanie w elemencie 1:

$\boxed{1}$

$$\underset{2\times L}{\{q\}_1} = [T_t]_1 \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ w_4 \end{Bmatrix}$$

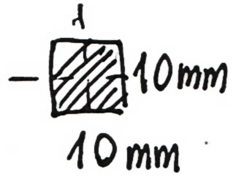
$$\varepsilon_1 = \frac{w_4 - 0}{l_1} = -\frac{10F}{3EA} = -0.25 \cdot 10^{-3}$$

$$\sigma_1 = E \cdot \varepsilon_1 = -\frac{10}{3}\frac{F}{A} = -50\,MPa \qquad N_1 = \sigma_1 \cdot A = -\frac{10}{3}F = -5000\,N$$

Ściskanie (możliwe wyboczenie?)

Siła krytyczna w elemencie 1:

$$P_{kr} = \frac{\pi^2 E J}{l_1^2}$$



$$J = \frac{(10mm)^4}{12}$$

$$P_{kr} = \frac{\pi^2 E \cdot 10^4 mm^4}{12 \cdot 400^2 mm^2} = \frac{\pi^2 \cdot 2 \cdot 10^5 \cdot 10^4 \; Nmm^2}{12 \cdot 16 \cdot 10^4 mm^2} = 10281 N$$

Współczynnik bezpieczeństwa $n = 2$

$$|N_1| < \frac{P_{kr}}{n}$$

Rozwiązanie w elemencie 2:

$$\{q\}_2 = \underset{2\times 6}{[\Gamma_t]_2} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{\sqrt{2}}{2} \cdot (u_4 + w_4) \end{Bmatrix}$$

$$\varepsilon_2 = \frac{\sqrt{2}}{2l_2}(u_4 + w_4) = \frac{\sqrt{2}}{2l_2} \cdot \left(\frac{(12l_2 + 10l_1)F}{3EA} - \frac{10Fl_1}{3EA} \right) = \frac{2\sqrt{2}F}{EA} = 0.212 \cdot 10^{-3}$$

$$\sigma_2 = E\varepsilon_2 = 42.43 \text{ MPa},$$

$$N_2 = \sigma_2 \cdot A = 4243 N$$

Rozwiązanie w elemencie 3:

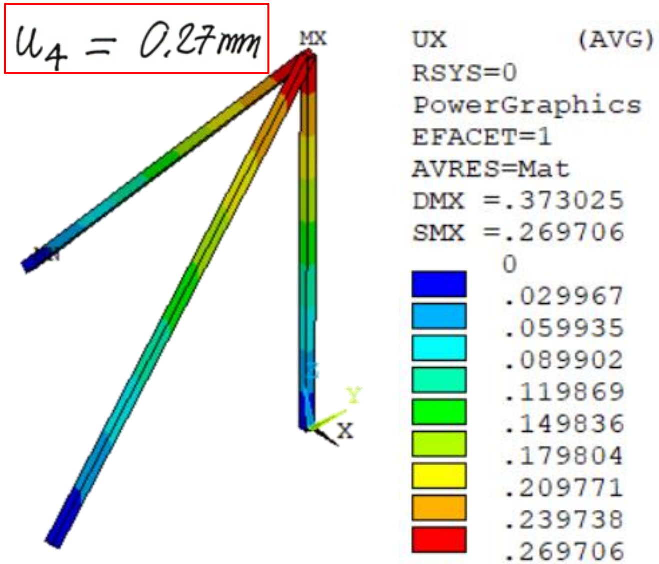
$$\{q\}_3 = [T_t]_3 \cdot \begin{Bmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.6 \cdot v_4 + 0.8 \cdot w_4 \end{Bmatrix}$$

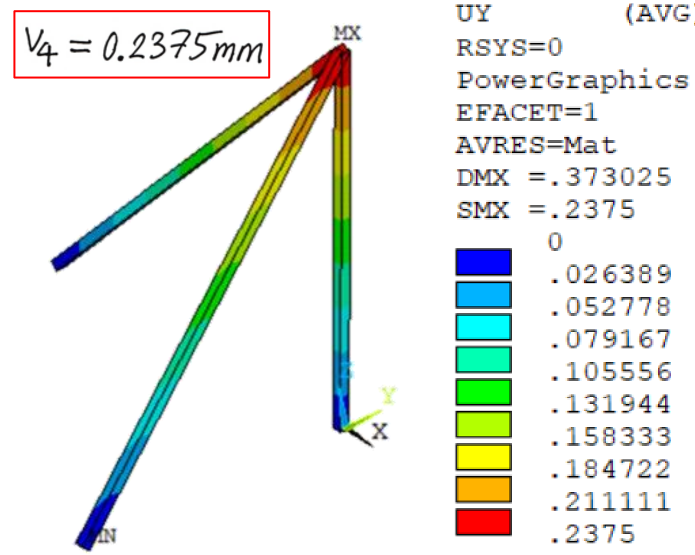
$$\varepsilon_3 = \frac{1}{l_3}\left(0.6 \cdot \frac{(100 l_3 + 160 l_1)F}{36 EA} - 0.8 \cdot \frac{10}{3}\frac{F l_1}{EA}\right) = \frac{5F}{3EA} = 0.125 \cdot 10^{-3}$$

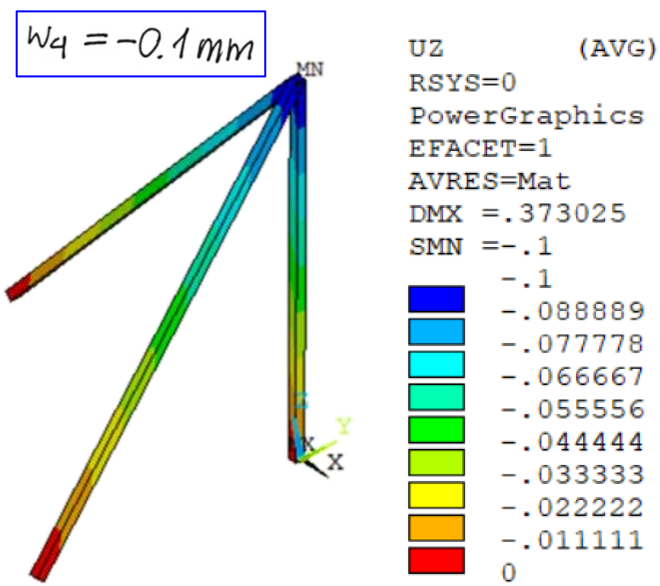
$$\sigma_3 = E \cdot \varepsilon_3 = 25 MPa$$

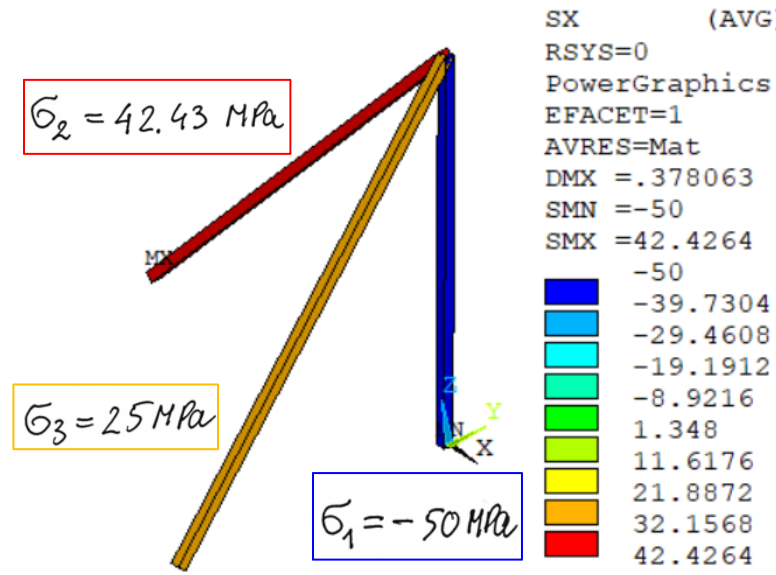
$$N_3 = \sigma_3 \cdot A = 2500 N$$

Porównanie wyników obliczeń z tymi uzyskanymi w programie ANSYS

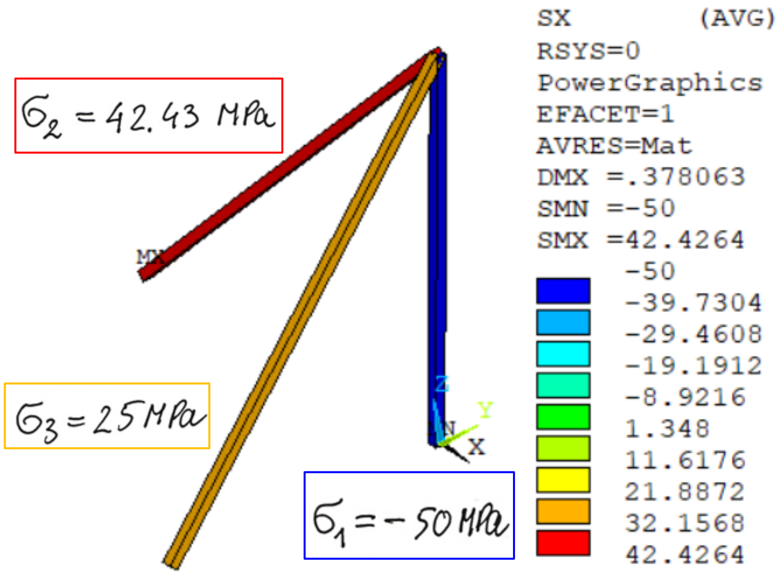


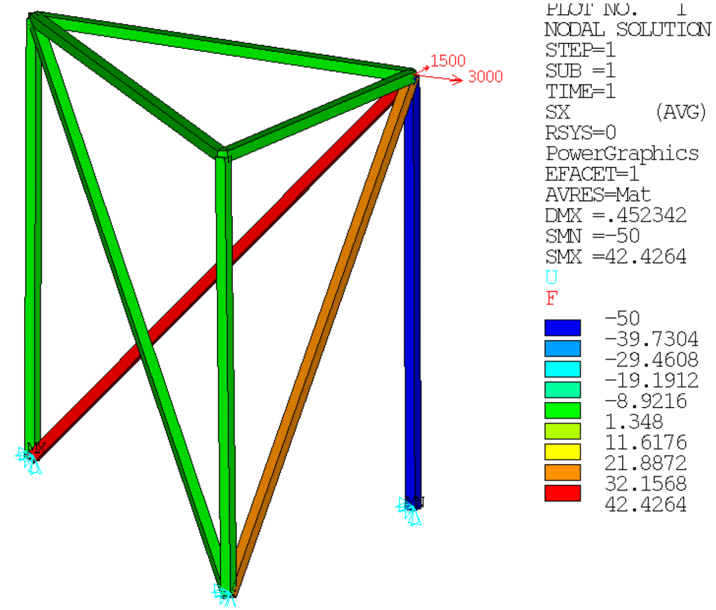


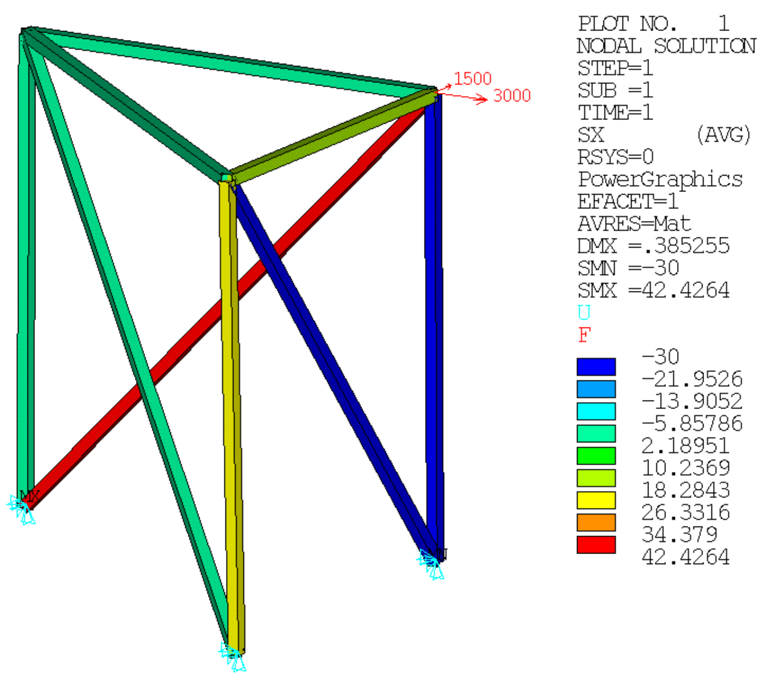


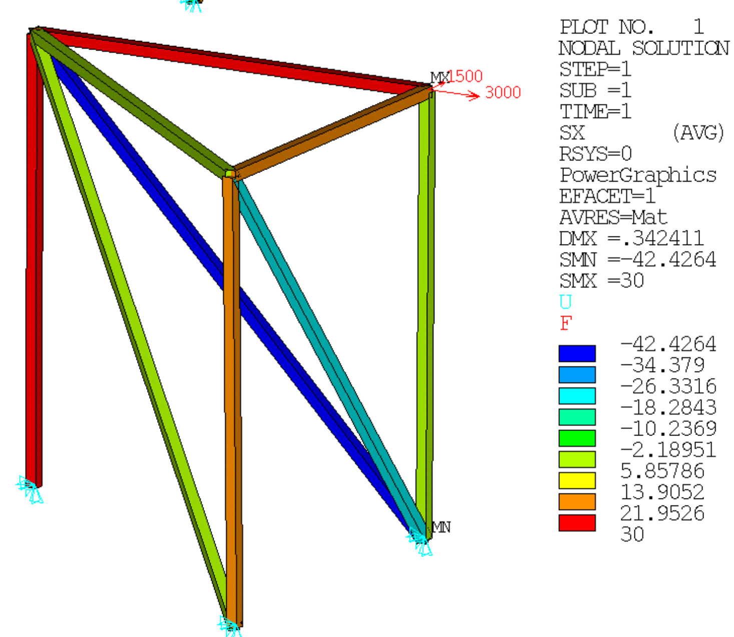


Modyfikacja modelu w programie ANSYS

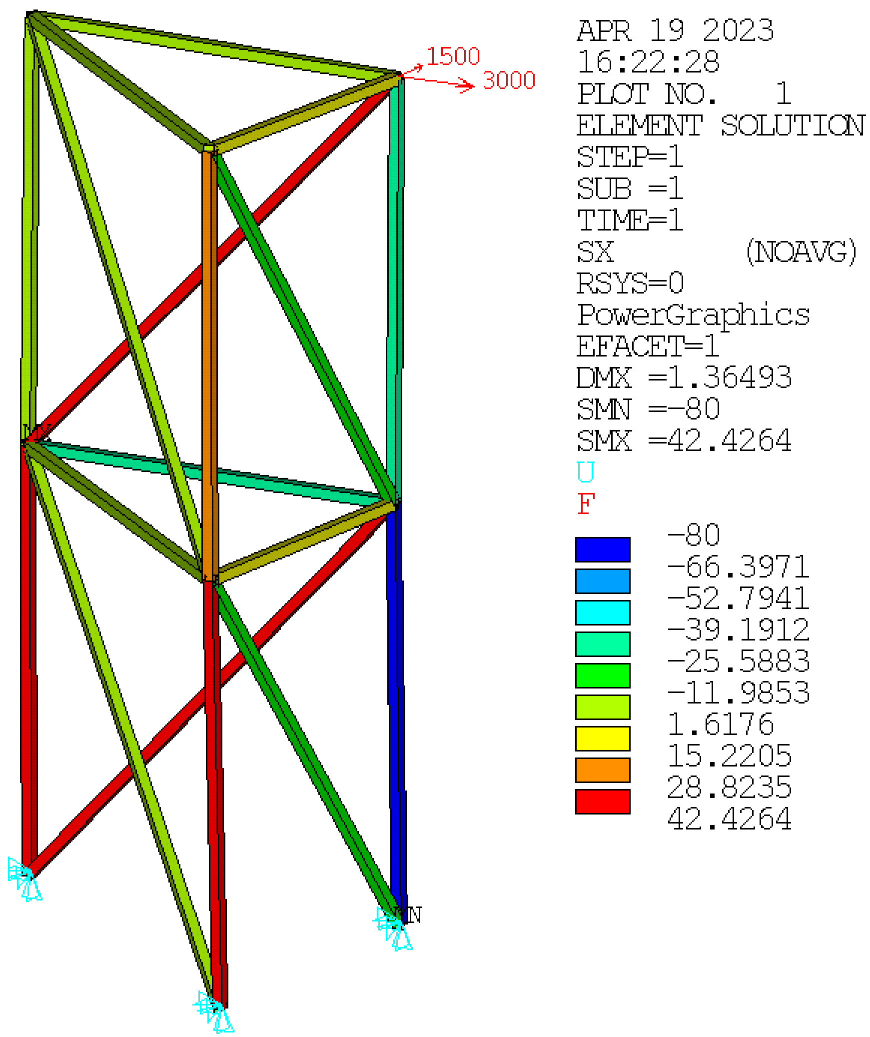


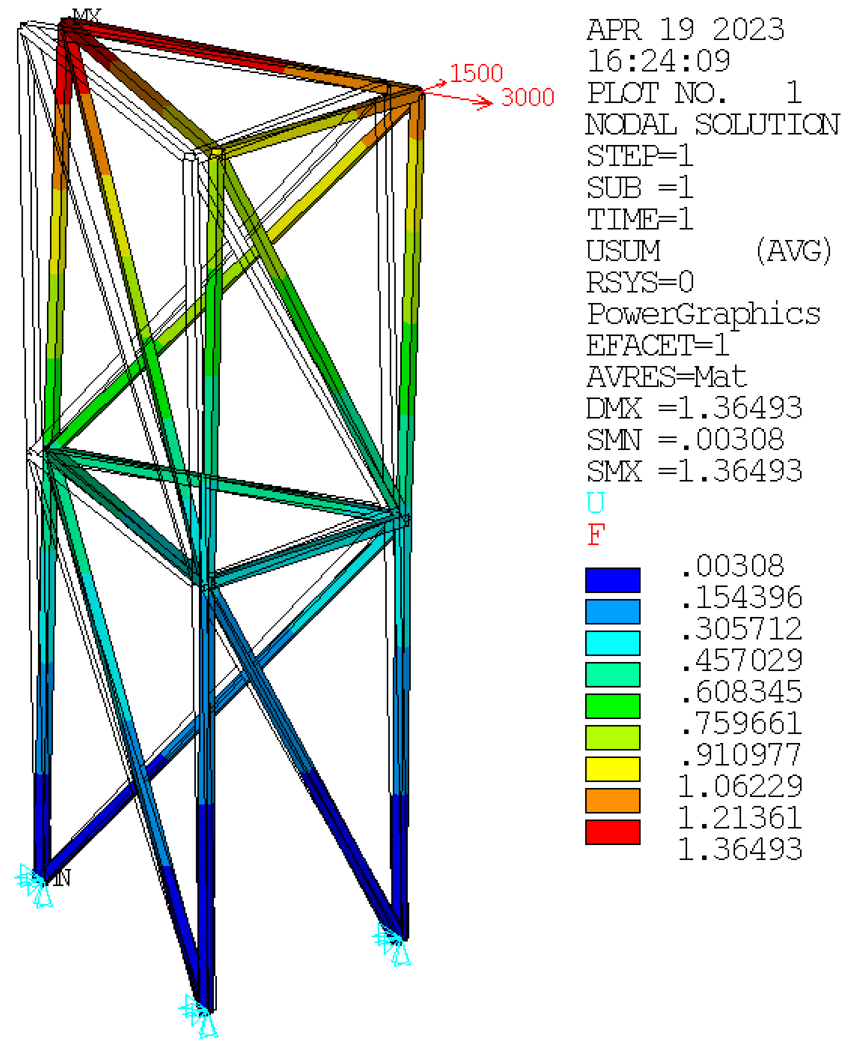




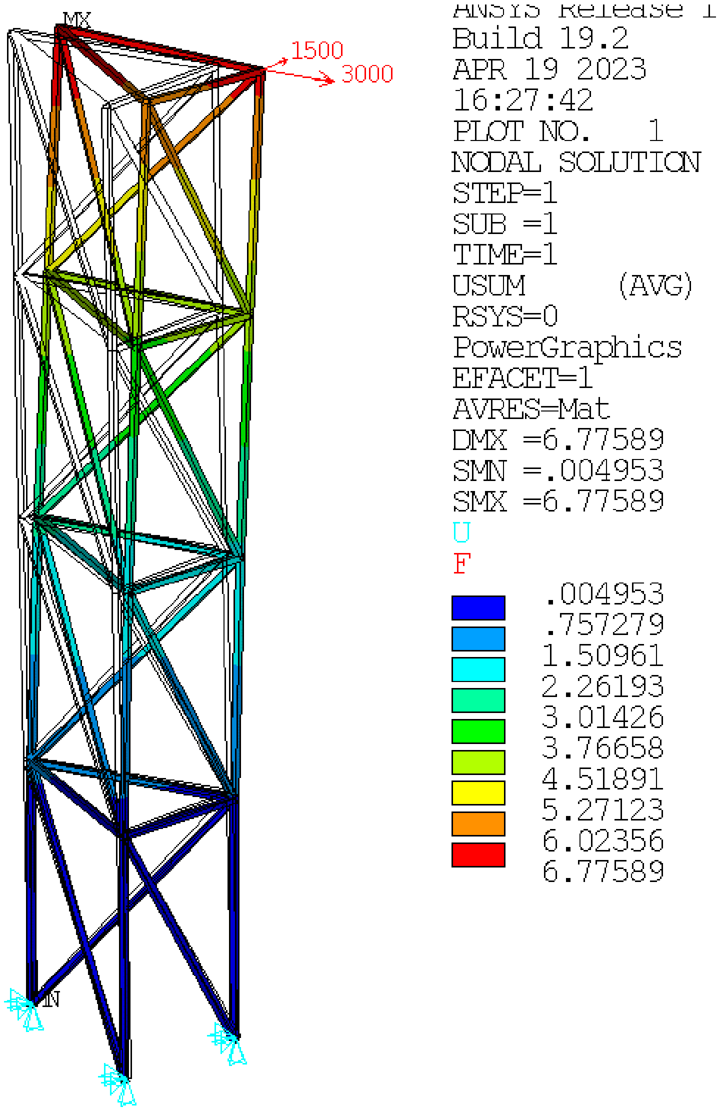


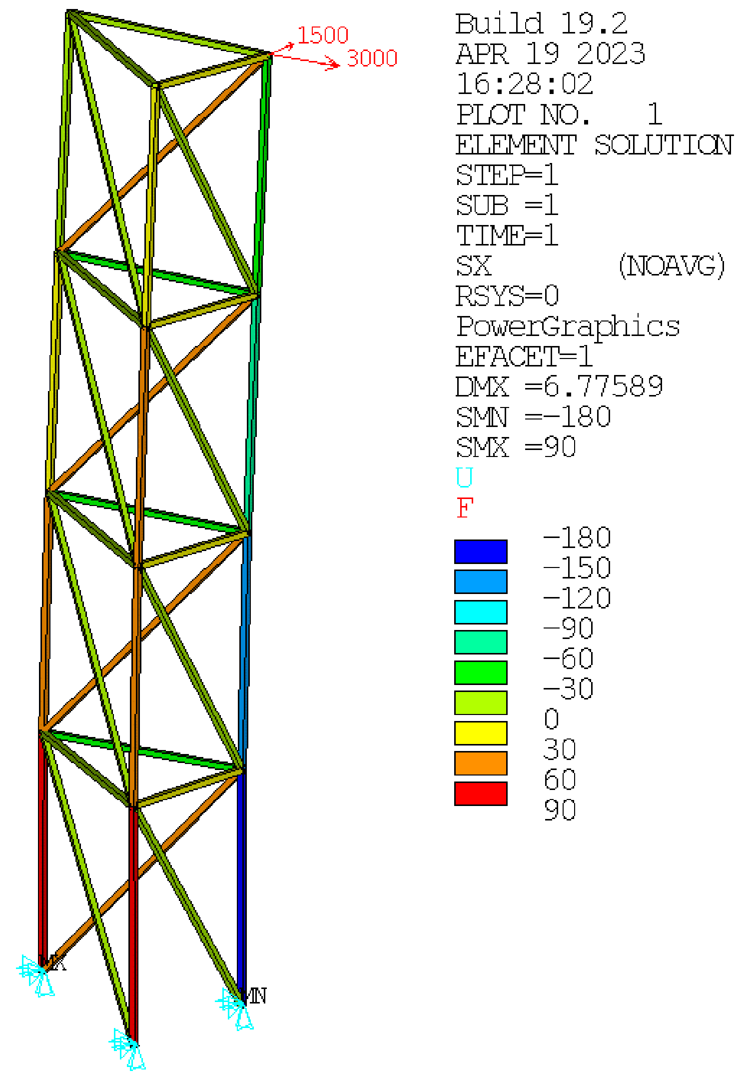
Rozbudowa modelu w programie ANSYS



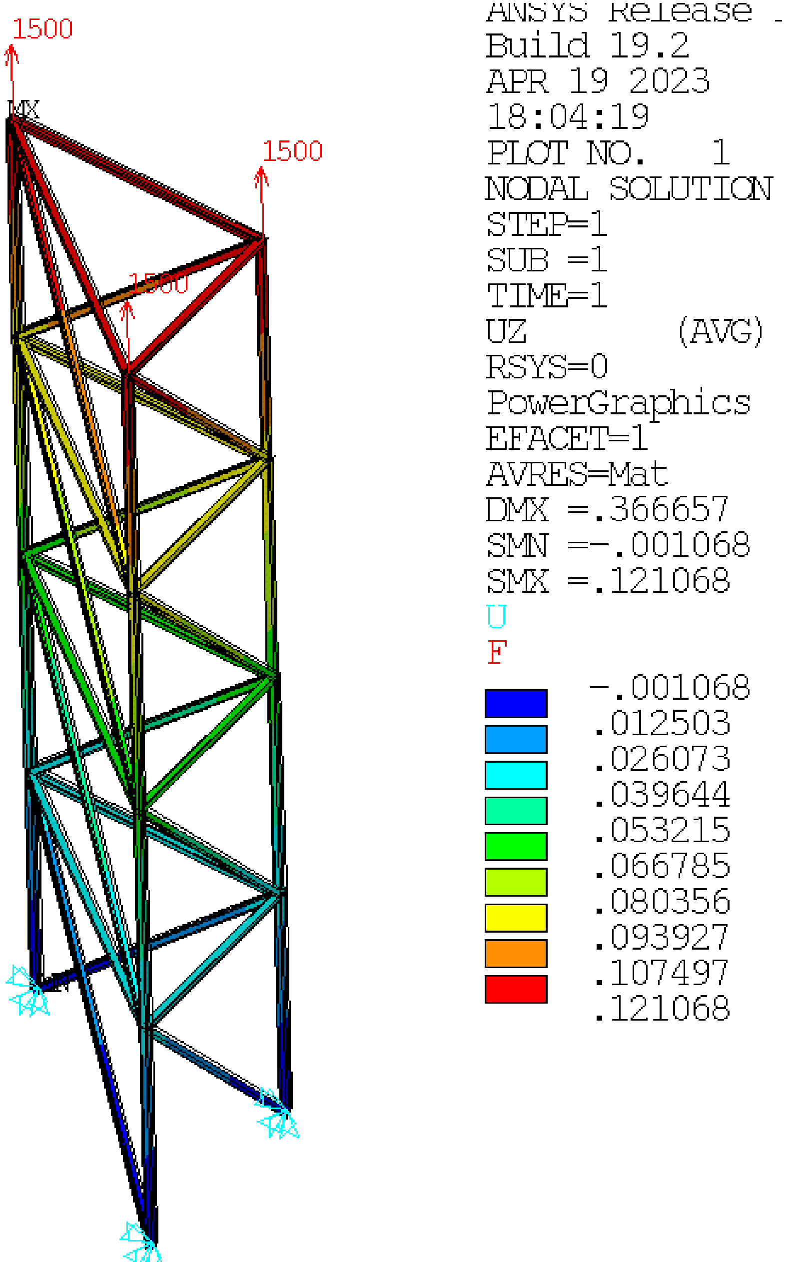


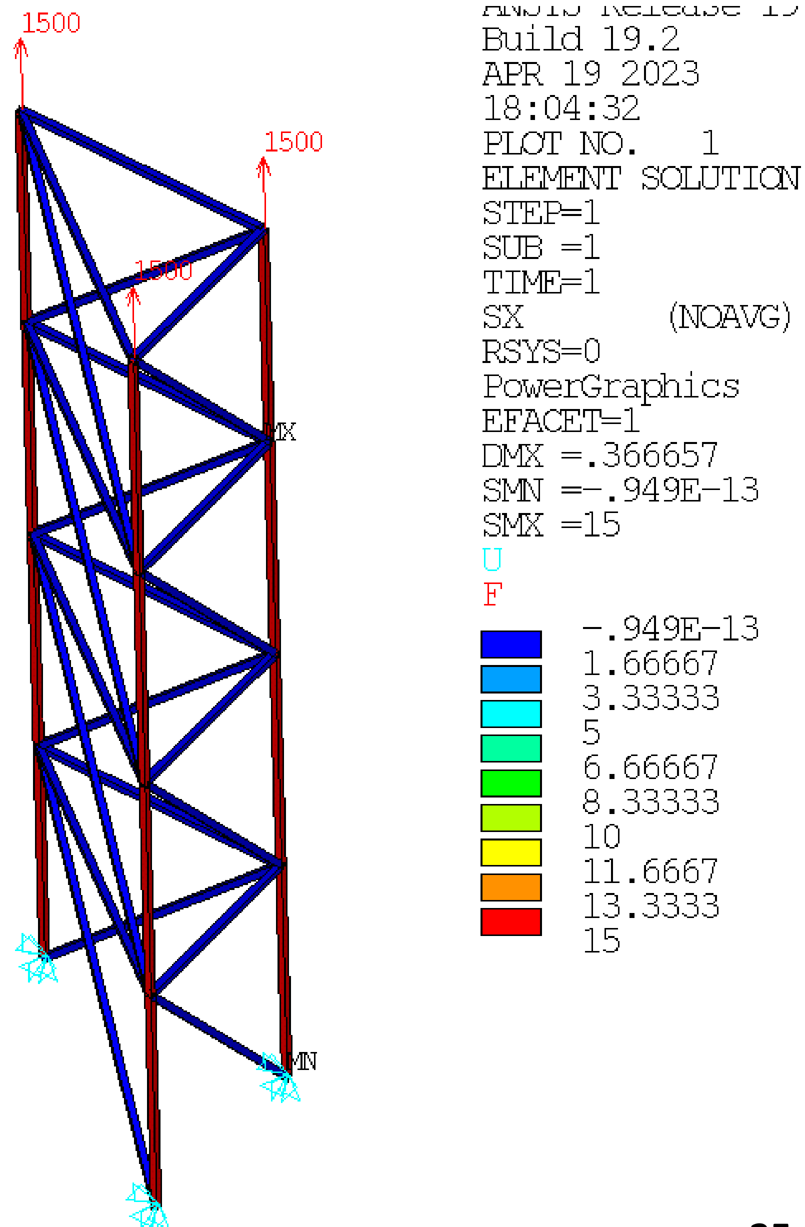
Rozbudowa modelu w programie ANSYS





Rozciąganie rozbudowanego modelu w programie ANSYS





Skręcanie rozbudowanego modelu w programie ANSYS